◎ Gauge fixed action (Faddeev-Popov):

- $$S' = S_0[A,\psi] + \int -\frac{1}{2\xi}(F_a)^2 + \bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b}\omega_b$$

or equivalently

- $$S = S_0[A,\psi] + \int \frac{\xi}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b}\omega_b$$

Becchi–Rouet–Stora
–Tyutin

BRST – Symmetry

- consider $\theta \equiv$ Grassmannian variable : $\delta_\theta \phi \equiv \theta\, s(\phi)$

BRST transformation

$$
\begin{cases}
\delta_\theta \psi = i\,\theta\,\omega_a T_a \psi \quad \leftarrow \alpha_a = \theta\,\omega_a \\
\delta_\theta A_{a\mu} = \theta\, D_\mu \omega_a = \theta\,\underbrace{\left[\partial_\mu \omega_a + f_{abc} A_{b\mu}\,\omega_c\right]}_{s(A)} \\
\delta \bar{\omega}_a = -\theta\, B_a = \theta\, s(\bar{\omega}_a) \\
\delta \omega_a = -\frac{1}{2}\theta\, f_{abc}\,\omega_b \omega_c \\
\delta B_a = 0
\end{cases}
$$

$\omega_1 \quad \omega_2 \qquad \omega_3$

$$(\omega_1 \omega_2) \cdot \omega_3 = (-1)^2 \omega_3 (\omega_1 \omega_2)$$

$\downarrow$

<u>boson</u>

- $S_0[A,\psi]$ obviously invariant : $\delta_\theta S_0 = 0$

- $\delta_\theta F_a(A) = \frac{\delta F_a}{\delta \alpha_b}\,\theta\,\omega_b = \theta\,\frac{\delta F_a}{\delta \alpha_b}\,\omega_b \equiv \theta\, s(F_a)$

$$\Rightarrow \delta_\theta \Big(\frac{\xi}{2} B_a^2 + B_a F_a + \underbrace{\bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b}\Big)$$

$$= 0 \quad \cancel{B_a \frac{\delta F_a}{\delta \alpha_b}\theta\,\omega_b} + \cancel{(-\theta B_a)\frac{\delta F_a}{\delta \alpha_b}}\,\omega_b + \bar{\omega}_a\, \delta_\theta\!\left(\frac{\delta F_a}{\delta \alpha_b}\omega_b\right)$$

$$= \bar{\omega}_a\, \delta_\theta\, s(F_a) = \bar{\omega}_a\, \theta\, s^2(F_a)$$

but as it turns out δ is nilpotent

$$\boxed{\delta^2 = 0}$$

and so $\delta_\theta S = 0$

Nilpotency of s

$[T_a, T_b] = i f_{abc} T_c$

- define $\omega = \omega_a T_a \qquad A_\mu = A^a_\mu T_a$
- $\delta_\theta \phi = \theta s \phi$

$$s\psi = i\omega_a T_a \psi$$
$$sA^a_\mu = D_\mu \omega_a = \partial_\mu \omega_a + f_{abc} A_{b\mu} \omega_c$$
$$s\bar{\omega}_a = -B_a$$
$$s\omega_a = \frac{1}{2} f_{abc} \omega_b \omega_c$$
$$sB_a = 0$$

$$s\psi = i\omega\psi$$
$$sA_\mu = D_\mu \omega$$
$$= \partial_\mu \omega - i[A_\mu, \omega]$$

$$s\omega = i\omega^2$$

$$\omega_a T_a \omega_b T_b = \frac{\omega_a \omega_b}{2}[T_a, T_b]$$

$\delta_\theta(\cdots) = \theta\, s(\cdots)$

$$\bullet\ \underbrace{\delta_\theta s\psi}\ = \theta s^2 \psi \quad = \delta_\theta(i\omega\psi) = i\theta\omega^2\psi + i\omega\theta\omega\psi$$

$$= i\theta(\omega^2 - \omega^2)\psi = 0$$

$$\bullet\ \underbrace{\delta_\theta s A_\mu}\ = \theta s^2 A_\mu \quad = \delta_\theta(D_\mu\omega) = \delta_\theta\left(\partial_\mu\omega - i[A_\mu, \omega]\right)$$

$$= \partial_\mu(i\theta\omega^2) - i[A_\mu, i\theta\omega^2] - i[\theta D_\mu\omega, \omega]$$

$$= i\theta\partial_\mu\omega^2 + [A_\mu, \theta\omega^2] - i\theta\partial_\mu\omega\,\omega + \underbrace{i\omega\theta}_{-i\theta\omega}\partial_\mu\omega - [\theta[A_\mu,\omega],\omega]$$

$$= i\theta\partial_\mu\omega^2 + [A_\mu, \theta\omega^2] - i\theta\partial_\mu\omega^2 - \theta(A_\mu\omega^2 - \omega A_\mu\omega) + \omega\theta(A_\mu\omega - \omega A_\mu)$$

$$= \theta[A_\mu, \omega^2] - \theta[A_\mu, \omega^2] = 0$$

Exercise!

- $\delta_\theta\, s(\bar{\omega}) = \theta\, s^2 \bar{\omega} = \delta_\theta(-B_e) = 0$

- $\delta_\theta\, s(B_e) = \theta\, s^2(B_e) = \delta_\theta(0) = 0$

- $\delta_\theta\, s\omega = \theta\, s^2\omega = \delta_\theta(i\omega^2) = i[i\theta\omega^2 \cdot \omega + i\omega\theta\omega^2]$

$\delta_\theta(i\omega\,\omega) = i\,\delta_\theta\omega\,\omega + i\omega\,\delta_\theta\omega$

$= i^2[\theta\omega^3 - \theta\omega^3] = 0$

$$\boxed{s^2 = 0 \quad \text{on elementary fields}}$$

- s^2 on product of fields

- $\underbrace{\delta_\theta \phi_1 \phi_2}_{\delta_\theta \phi_1 \phi_2 + \phi_1 \delta_\theta \phi_2} = \theta s\phi_1 \phi_2 + \phi_1 \theta s\phi_2 = \theta\left(s\phi_1 \phi_2 + (-)^{F_1} \phi_1 s\phi_2\right)$

$$\Longrightarrow \boxed{s(\phi_1 \phi_2) = s\phi_1 \phi_2 + (-)^{F_1} \phi_1 s\phi_2}$$

- $s^2 \phi_1 \phi_2 = s\left(s\phi_1 \phi_2 + (-)^{F_1} \phi_1 s\phi_2\right) =$

$$= s^2\phi_1 \phi_2 + (-)^{F_1+1} s\phi_1 s\phi_2 + (-1)^{F_1} s\phi_1 s\phi_2 + (-1)^{2F_1} \phi_1 s^2 \phi_2$$

$$= s^2\phi_1 \phi_2 + \phi_1 s^2 \phi_2 = 0$$

- $s^2(\phi_1 \cdots \phi_n) = 0$ by induction

$$\Longrightarrow \quad \forall \quad Y[\Phi] \qquad \boxed{\delta^2 Y[\Phi] = 0}$$

- back to $S = S_0 + \underbrace{\int \frac{\xi}{2} B^2 + BF + \bar{\omega} \frac{\delta F}{\delta \alpha} \omega}$

$$\frac{\xi}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b =$$

$$= - \delta \left[\frac{\xi}{2} \bar{\omega}_a B_a + \bar{\omega}_a F_a \right] \equiv \delta Y$$

$$= - (\delta \bar{\omega}_a) \left(\frac{\xi}{2} B_a + F_a \right) + \bar{\omega}_a \delta F_a$$

$$S = \underbrace{S_0[A,\psi]}_{} + \underbrace{s\, Y[A,\dots]}_{}$$

· non-trivially BRST invariant

"BRST exact" $\equiv$ trivially BRST inv.

- $s S_0 = 0$ but
- $S_0 \neq s(\dots)$

---

differential form calculus analysis

space $X \longrightarrow A_\mu \, dx^\mu = A \equiv$ differential form

- $dA \equiv \partial_\nu A_\mu \, dx^\mu \wedge dx^\nu = (\partial_\nu A_\mu - \partial_\mu A_\nu) \frac{1}{2} dx^\mu \wedge dx^\nu$

$$\delta \longleftrightarrow \vec{\nabla}\wedge$$

$$S = S_0 + \delta Y \qquad \vec{A} = \vec{A}_0 + \vec{\nabla} Y$$

$$\delta S = 0 \qquad \vec{\nabla}\wedge\vec{A} = \vec{\nabla}\wedge\vec{A}_0 + \vec{\nabla}\wedge\vec{\nabla} Y = 0$$

$$\delta S_0 = 0 \qquad \vec{\nabla}\wedge\vec{A}_0 = 0$$

$$S_0 \neq \delta(\cdots) \qquad \vec{A}_0 \neq \vec{\nabla}(\cdots)$$

A_i

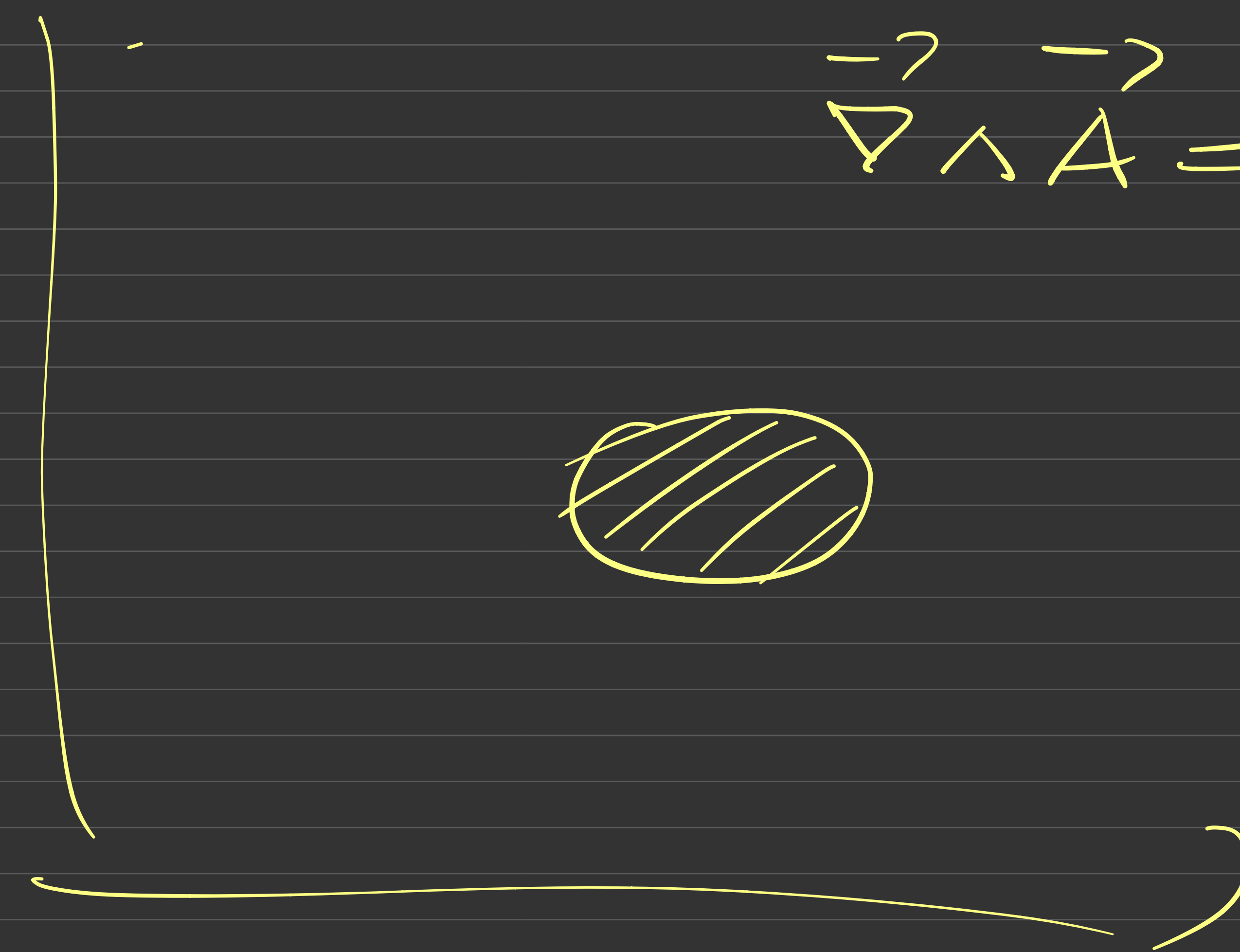
$$\nabla \times A = 0 \begin{cases} \xrightarrow{\text{trivial}} \vec{A} = \vec{\nabla}\phi & \vec{A} \equiv \text{exact} \\ \xrightarrow{\text{non-trivial}} \vec{A} \neq \vec{\nabla}\phi & \vec{A} = \text{closed but not exact} \end{cases}$$

Ex

Aharonov-Bohm

$$X = \mathbb{R}^2 - D$$

$$\vec{\nabla} \wedge \vec{A} = 0$$



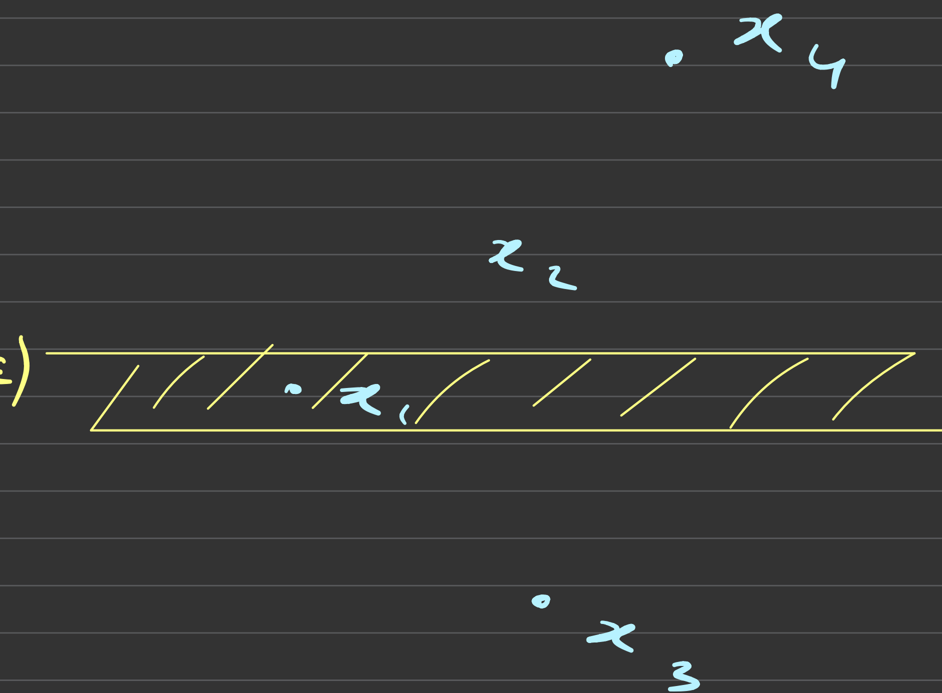
- $$i\,\partial_\mu \langle T(J^\mu_{BRST}(y)\,\phi_1(x_1)\cdots\phi_n(x_n))\rangle =$$

$$=\delta(y-x_1)\langle T(\delta\phi_1(x_1)\cdots\phi_n(x_n))\rangle + \delta(y-x_2)(-)^{F_1}\langle T(\phi_1\,\delta\phi_2\cdots\phi_n)\rangle +$$

$$\cdots + \delta(y-x_n)(-)^{F_1+\cdots F_{n-1}}\langle T(\phi_1\cdots\phi_{n-1}\,\delta\phi_n)\rangle$$

- $$Q \equiv \int J^0_{BRST}(x,t)\,d^3x \longrightarrow t\text{-indep}$$

$$\int_{t_1-\varepsilon}^{t_1+\varepsilon} \partial_0 J^0 = J^0(t_1+\varepsilon) - J^0(t_1-\varepsilon)$$



- $$\int_{t_1-\varepsilon}^{t_1+\varepsilon} dy^0\, d^3y \Longrightarrow$$

$$\langle 0|T(i[Q,\phi_1\}_\pm\,\phi_2\cdots\phi_n)|0\rangle = \langle 0|T(\delta\phi_1\,\phi_2\cdots\phi_n)|0\rangle$$

$\triangle$ Physical states $\quad |\psi_i(t_i)\rangle \equiv e^{iHt_i}|\psi_i\rangle$

$$\int \mathcal{D}\phi \; e^{i(S_0+\delta Y)} \, \psi_f^*(\phi_f)\,\psi_i(\phi_i) = \langle\psi_f| e^{-iH(t_f-t_i)} |\psi_i\rangle$$

$$\equiv \langle \psi_f(t_f) | \psi_i(t_i) \rangle$$

- $Y \to Y + \Delta Y \quad \Longrightarrow$

$$\delta\left(\langle\psi_f|\psi_i\rangle\right) \equiv \int \mathcal{D}\phi \; i\,\delta\Delta Y \, e^{i(S_0+\delta Y)} \psi_f^* \psi_i = i\langle\psi_f|[Q,\Delta Y]|\psi_i\rangle$$

- $\langle\psi_f(t_f)|\psi_i(t_i)\rangle$ should be independent of Y choice

QM IV

$$K(x_f, t_f; x_i, t_i) = \int_{\substack{x=x_i \; t=t_i \\ x=x_f \; t=t_f}} \mathcal{D}x \; e^{iS[x]} = \langle x_f | e^{-iH(t_f-t_i)} | x_i \rangle$$

$$= \langle \psi_f | e^{-iH(t_f-t_i)} | \psi_i \rangle = \int dx_f \, dx_i \, \langle x_f | e^{-iH(t_f-t_i)} | x_i \rangle \, \psi_f^*(x_f) \psi_i(x_i)$$

$$= \int \mathcal{D}x \; e^{iS[x]} \; \psi_f^*(x_f) \psi_i(x_i)$$

$$\langle \psi_f Q | \Delta Y | \psi_i \rangle + \langle \psi_f | \Delta Y | Q \psi_i \rangle = 0$$

$\Longrightarrow$ physical $\psi_{i,f} \Rightarrow \langle \psi_f | [Q, \Delta Y]_+ | \psi_i \rangle = 0 \quad \forall \Delta Y$

$$\boxed{Q|\psi_i\rangle = \langle \psi_f | Q = 0}$$

- physical Hilbert space $\subset \mathrm{Ker}_Q$ $\qquad \mathrm{Ker}_Q$
- physical Hilbert space $= \mathrm{Ker}_Q$?
- " " $= \mathrm{Ker}_Q / \mathrm{Im}_Q$ $\qquad \mathrm{Im}_Q \subset \mathrm{Ker}_Q$

Physical quantities $\Longleftrightarrow$ BRST cohomology

- Q conserved
- $Q^2 = 0$

- dynamics: $S = S_0 + \delta Y$

$S_0 \rightarrow$ Q-closed:

- $\delta S_0 = 0$
- $S_0 \neq \delta(\cdots)$

$\delta Y \rightarrow$ Q-exact

- cohomology: $S \sim S + \delta \Delta Y$

$$\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}B\,\mathcal{D}\omega\,\mathcal{D}\bar{\omega}\; e^{iS_0 + gY}$$

$$\downarrow$$

$$B,\ \omega,\ \bar{\omega},\ \partial_\mu A^\mu$$

$$\int \mathcal{D}\omega\; f(\omega) \equiv \frac{\partial}{\partial \omega} f(\omega)$$

- **states** : $|\psi\rangle$ • $Q|\psi\rangle = 0$

  • Q-closed : $|\psi\rangle \neq Q|\varphi\rangle$

  • Q-exact : $|\psi\rangle = Q|\varphi\rangle$ $\langle\psi|\psi\rangle = 0$

- cohomology $|\psi\rangle \sim |\psi\rangle + Q|\varphi\rangle \equiv |\psi'\rangle$

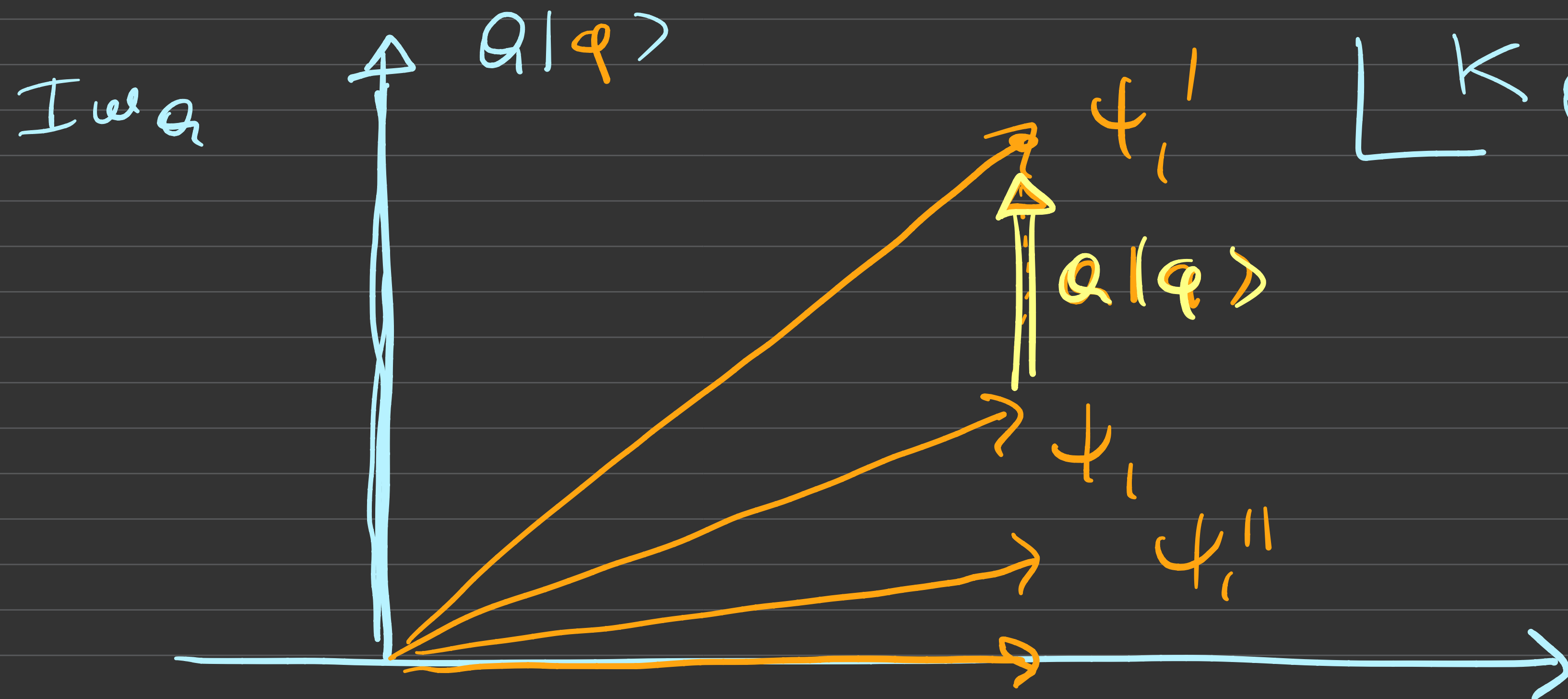
• $\langle\psi'|\psi'\rangle = \left(\langle\psi| + \langle\varphi|Q\right)\left(|\psi\rangle + Q|\varphi\rangle\right) = \langle\psi|\psi\rangle$

• $\langle\psi'|\hat{O}|\psi'\rangle = \langle\psi|\hat{O}|\psi\rangle$ if $[Q, \hat{O}]_{\pm} = 0$

• $e^{-iHt}|\psi'\rangle = e^{-iHt}|\psi\rangle + \underbrace{e^{-iHt}Q|\varphi\rangle} \sim e^{-iHt}|\psi\rangle$

$$+ Q\, e^{-iHt} |q\rangle$$

physical Hilbert space $=$ $\mathrm{Ker}\, Q$ modulo $\mathrm{Im}\, Q$



$\llcorner\, \mathrm{Ker}\, Q$

$\mathrm{Im}_Q \subset \mathrm{Ker}_Q$

physical Hilbert space $\equiv \mathrm{Ker}\, Q / \mathrm{Im}\, Q$

physics sits in the <u>BRST cohomology</u>

Gupta-Bleuler

projetion

- $Q|\psi\rangle = 0$

$L(k)|\psi\rangle = 0$

$L(k) = k^\mu Q_\mu(k)$

cohomology

- $|\psi\rangle \sim |\psi\rangle + Q|\varphi\rangle$

$|\psi\rangle \sim |\psi\rangle + L|\varphi\rangle$

$$|\psi\rangle = a|a\rangle$$

$$\langle\psi|\psi\rangle = 0$$

$$\langle\psi|\hat{O}|\psi\rangle = 0 \qquad \langle a| a\, \hat{O}\, a|a\rangle$$

$$|\psi\rangle \approx 0$$